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INVERSE PROBLEMS IN RADIATIVE TRANSFER: I AYERED MEDIA

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The RAND Corporation
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INVERSE PROBLEMS IN RADIATIVE TRANSFER: LAYERED MEDIA

R. E. Bellman, H. H. Kagiwada, R. E. Kalaba and S. Ueno

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PREFACE

A basic problem in radiative transfer is the estimation of the physical parameters of a scattering and absorbing medium based on measurements of the diffusely reflected light. In this series of papers we shall show how such tasks may be viewed as nonlinear boundary-value problems which may be solved computationally using the technique of quasilinearization.

In this Memorandum we consider a stratified medium consisting of two layers. Our aim is to determine the optical depth and the albedo for single scattering of each layer based on measurements of the angular dependence of the light diffusely reflected from the slab. The results of some numerical experiments are presented, and FORTRAN IV listings are provided.

The results will be of interest to meteorologists, experimental physicists, and control engineers.

SUMMARY

Consider a slab which consists of two layers which absorb and scatter light. First the basic differential-integral equation for the intensity of the diffusely reflected light is given, the source being uniform parallel rays. Then we show how to determine the nature of each layer using measurements of the light diffusely reflected from the slab. We view this as a nonlinear boundary-value problem and show that it may be resolved computationally using the technique of quasilinearization. Numerical examples and FORTRAN programs are provided.

I. INTRODUCTION

Problems of radiative transfer in planetary and stellar atmospheres have been extensively treated by many investigators.[1,2,3] Much of this work has dealt with what we may call direct problems, i.e., the determination of the intensities of radiation produced by certain light incident on a medium with known scattering and absorbing properties. Recently, though, as a growing list of papers attests, there has been increasing interest in inverse problems. In these, the aim is to deduce the nature of the medium (and often the source) on the basis of measurements of the radiation field produced by incident radiation.[4,5,6]

With this Memorandum we initiate the study of inverse problems in radiative transfer via the method of quasilinearization.[7,8] We wish to show that such problems can be viewed as multipoint boundary-value problems for large systems of nonlinear ordinary differential equations. These equations can then be resolved numerically using current digital computers and the method of quasilinearization.

We consider a slab which consists of two layers of isotropically scattering material of unknown optical thickness and albedo for single scattering. Given the measured intensity of the light diffusely reflected from the slab, we wish to determine the optical thickness of the two layers and their albedos for single scattering.

In subsequent papers we shall consider more complex processes involving anisotropic scattering, time-dependence, and noncoherent scattering.

II. BASIC EQUATIONS

We consider an inhomogeneous, plane-parallel, non-emitting and isotropically scattering atmosphere of finite optical thickness τ_1 whose optical properties depend only upon τ, the optical height above the bottom ($0 \leq \tau \leq \tau_1$). Let parallel rays of light of net flux π per unit area normal to their direction of propagation be incident on the upper surface in the direction characterized by the number μ_o ($0 < \mu_o < 1$), where μ_o is the cosine of the angle measured from the normal to the surface. The bottom surface is a completely absorbing barrier, so that no light is reflected from it. This is not an essential requirement. See Fig. 1.

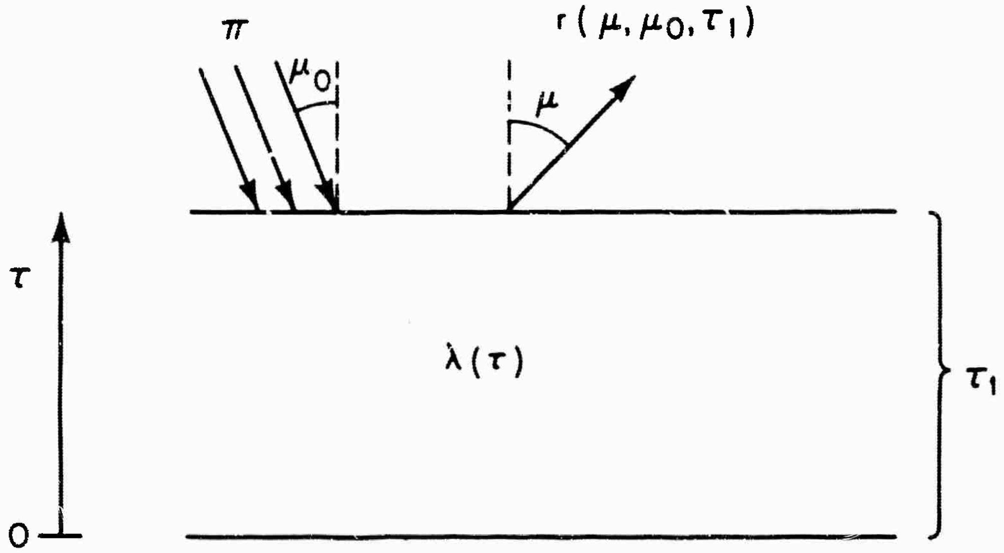

Fig. 1—Incident and reflected rays for a slab of thickness τ_1

The intensity of the diffusely reflected light in the direction $\cos^{-1}\mu$ is $r(\mu, \mu_o, \tau_1)$. We define a related function $R(\mu, \mu_o, \tau_1)$, symmetric in μ and μ_o, by writing

$$r(\mu, \mu_o, \tau_1) = \frac{R(\mu, \mu_o, \tau_1)}{4\,\mu} \,. \tag{1}$$

The function R satisfies the integro-differential equation(1,9)

$$\begin{aligned}\frac{\partial R(\mu,\mu_o,\tau_1)}{\partial \tau_1} &= -\left(\frac{1}{\mu}+\frac{1}{\mu_o}\right)R(\mu,\mu_o,\tau_1)\\ &+ \lambda(\tau_1)\left[1+\frac{1}{2}\int_0^1 R(\mu,\mu',\tau_1)\frac{d\mu'}{\mu'}\right]\left[1+\frac{1}{2}\int_0^1 R(\mu'',\mu_o,\tau_1)\frac{d\mu''}{\mu''}\right]\end{aligned} \tag{2}$$

where $\lambda(\tau_1)$ is the albedo for single scattering, and R is subject to the condition

$$R(\mu,\mu_o,0) = 0. \tag{3}$$

This equation is obtained by means of the theory of invariant imbedding. A discrete version of Eqs. (2) and (3) is obtained by replacing the integrals by finite sums using a Gaussian quadrature formula of order N. Let us introduce new functions $R_{ij}(\tau_1)$,

$$R_{ij}(\tau_1) = R(\mu_i,\mu_j,\tau_1); \quad i,j = 1,2,\ldots,N, \tag{4}$$

where $\{\mu_i\}$ is the set of N roots of the shifted Legendre polynomial $P_N^*(\mu) = P_N(1-2\mu)$. The values of μ_i and the corresponding Christoffel weights are tabulated in Ref. 10 for N = 3,4,...,15. The functions $R_{ij}(\tau_1)$ satisfy the system of ordinary nonlinear differential equations

$$\frac{dR_{ij}(\tau_i)}{d\tau_1} = -\left(\frac{1}{\mu_i} + \frac{1}{\mu_j}\right) R_{ij}(\tau_1)$$

$$+ \lambda(\tau_i) \left[1 + \frac{1}{2} \sum_{k=1}^{N} R_{ik}(\tau_1) \frac{w_k}{\mu_k}\right] \left[1 + \frac{1}{2} \sum_{k=1}^{N} R_{kj}(\tau_1) \frac{w_k}{\mu_k}\right] , \tag{5}$$

with the initial conditions

$$R_{ij}(0) = 0 , \tag{6}$$

for $i = 1,2,\ldots,N$; $j = 1,2,\ldots,N$.

III. AN INVERSE PROBLEM

Let us now consider a medium composed of two layers, with albedo for single scattering λ_1 in the lower layer and albedo λ_2 in the upper layer ($\lambda_1 \neq \lambda_2$). The total optical thickness of this slab is T and the thickness of the lower layer is c. See Fig. 2. We wish to determine c, T, and the parameters λ_1 and λ_2.

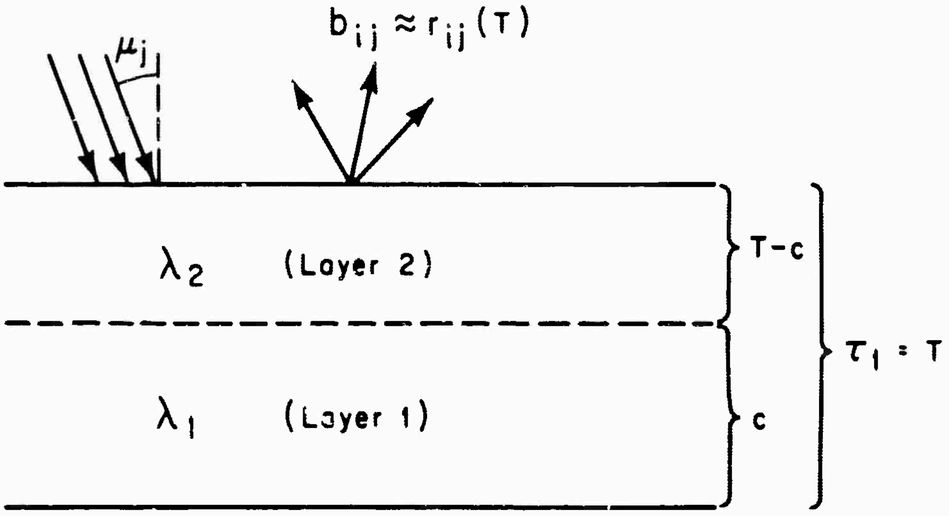

Fig. 2—A stratified medium

Let us assume that we have obtained N^2 (noisy) measurements of the diffusely reflected light, b_{ij}, where b_{ij} is the intensity of the light diffusely reflected in the direction μ_i caused by incident parallel rays of net flux π in the direction μ_j. The constants which characterize this medium, λ_1,λ_2,c, and T are to be determined so that the theoretical diffuse reflection pattern produced by using the estimated values in the differential Eqs. (5) will agree as closely as possible, in the least squares sense, with the observed field $\{b_{ij}\}$, i.e., we wish to minimize the expression

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \left[r_{ij}(T) - b_{ij}\right]^2. \tag{7}$$

Let us assume that the albedo function has the form

$$\lambda(\tau) = a + b \tanh 10 (\tau - c) \tag{8}$$

so that

$$\begin{aligned} \lambda_1 &\cong a - b \qquad \text{in Layer 1,} \\ \lambda_2 &\cong a + b \qquad \text{in Layer 2,} \end{aligned} \tag{9}$$

and c is the position of the boundary between Layer 1 and Layer 2. See Fig. 3.

The "observations" $\{b_{ij}\}$ are produced computationally with the use of Eq. (8) for the albedo function, where we set

$$a = 0.5,\ b = 0.1,\ c = 0.5, \tag{10}$$

and with the use of the differential Eqs. (5), integrating out to a thickness

$$T = 1.0\ .$$

The observations which this produces are given in Table 1.

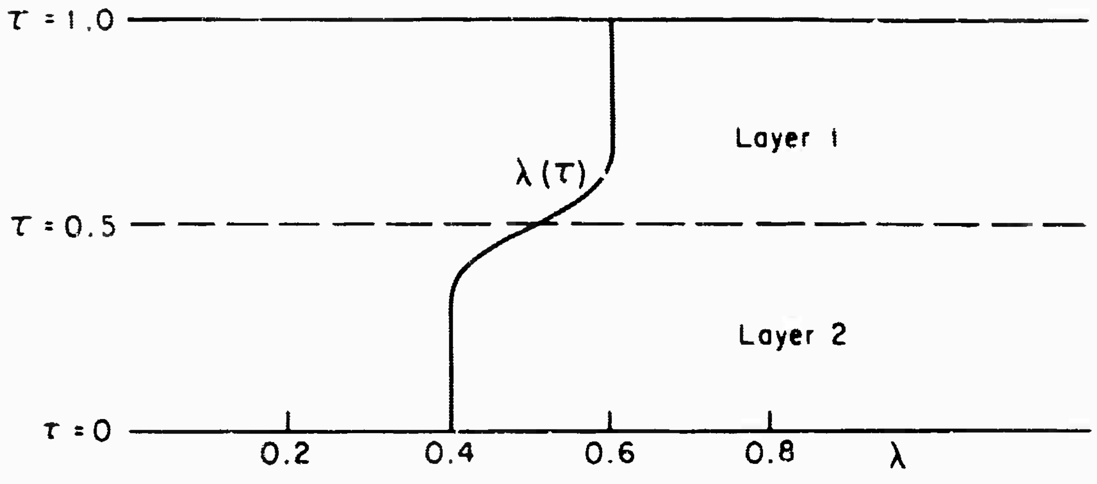

Fig. 3—The albedo function

Table 1

THE MEASUREMENTS $\{b_{ij}\}$

	j = 1	2	3	4	5	6	7
i = 1	0.079914	0.028164	0.014304	0.009104	0.006707	0.005515	0.004970
2	0.143038	0.091522	0.058437	0.040826	0.031405	0.026378	0.023989
3	0.167000	0.134331	0.099653	0.075106	0.060044	0.051445	0.047248
4	0.178898	0.157955	0.126408	0.099392	0.081253	0.070435	0.065042
5	0.185284	0.170817	0.142072	0.114229	0.094495	0.082423	0.076332
6	0.188723	0.177733	0.150791	0.122665	0.102104	0.089349	0.082870
7	0.190354	0.180898	0.154995	0.126773	0.105829	0.092748	0.086083

IV. NUMERICAL SOLUTION VIA QUASILINEARIZATION(7,8,11,12)

Let us consider the problem posed above in more general and flexible terms. An R-dimensional column vector x(t) is a solution of the differential equation

$$\dot{x} = f(x) , \quad 0 \leq t \leq T . \tag{11}$$

The value of T is assumed known. This again represents no essential restriction as we shall see. We wish to determine the initial vector c,

$$x(0) = c , \tag{12}$$

in such a manner that we minimize the quadratic form

$$(x(T) - b, x(T) - b) = Q , \tag{13}$$

where x(T) is the solution of Eq. (1) for t = T. The first s components of c are required to be zero and the remaining ones are variables to be determined.

Our computational formalism proceeds as follows: First an initial approximation to the desired initial vector is selected. We then proceed inductively. Suppose that we have obtained a kth approximation to c, which we denote by c^k, and a kth approximation to the solution function x(t), which we denote by $x^k(t)$. Note that the superscripts refer to the order of the approximation and not to the components of c and x. We obtain the vectors c^{k+1} and $x^{k+1}(t)$ in this manner. The vector $x^{k+1}(t)$ is a solution of the linear system in x^{k+1}

$$\dot{x}^{k+1} = f(x^k) + J(x^k)\,(x^{k+1} - x^k) , \tag{14}$$

where $J(x^k)$ is the Jacobian matrix with elements

$$J_{ij} = (\partial f_i / \partial x_j) \ . \tag{15}$$

We produce a particular solution $p(t)$ of the system (14) by numerically integrating the system

$$\dot{p} = f(x^k) + J(x^k)\ (p - x^k) \ , \tag{16}$$

where $p(t)$ is subject to the initial condition

$$p(0) = 0 \ . \tag{17}$$

Then we produce numerically R-s independent solutions of the homogeneous system

$$\dot{h}_i = J(x^k) h_i \ , \tag{18}$$

where the vector $h_i(t)$ is subject to the initial condition

$$(j^{th} \text{ component of } h_i(0)) = \delta_{ij} \ , \quad i = s + 1, \ldots, R \ , \tag{19}$$

δ_{ij} being the Kronecker delta function. These integrations are readily carried out on the interval $0 \le t \le T$, since complete sets of initial conditions are given. From general theory we know that the vector x^{k+1} is representable in the form

$$x^{k+1}(t) = p(t) + \sum_{i=s+1}^{R} m_i h_i(t) \ , \tag{20}$$

where the numbers m_i are arbitrary constants. In view of the way we have chosen the initial conditions in Eqs. (17) and (19), we see that

m_i represents the initial value of the i^{th} component of the vector x^{k+1}. In particular, when the integrations are completed, we shall have, in numerical form, the values of the vectors p(T), $h_{s+1}(T)$, ..., $h_R(T)$.

We now wish to minimize the form

$$\left(p(T) + \sum_{i=s+1}^{R} m_i h_i(T) - b,\ p(T) + \sum_{i=s+1}^{R} m_i h_i(T) - b\right) \tag{21}$$

$$= F\left(m_{s+1}, m_{s+2}, \ldots, m_R\right).$$

over the values of m_i, $i = s+1, s+2, \ldots, R$, where b is known experimentally and p(T), and $h_{s+1}(T)$, $h_{s+2}(T)$, ..., $h_R(T)$ are known computationally. This minimization is done by solving, numerically, the linear algebraic system of equations

$$\partial F/\partial m_i = 0, \qquad i = s+1, s+2, \ldots, R\ . \tag{22}$$

The values of m_{s+1}, m_{s+2}, ..., m_R so obtained are the values of the last R-s components of the vector c^{k+1}, and $x^{k+1}(t)$ can be determined from Eq. (20) using the values of m_{s+1}, m_{s+2}, ..., m_R and the stored values of p(t) and $h_i(t)$, $i = s+1, s+2, \ldots, R$. The entire process is then repeated to obtain the $(k+2)^{nd}$ approximations.

To simplify the procedure of finding the m's, let us observe that the relation

$$x^{k+1}\ (T) \cong b \tag{23}$$

can be written in the form

$$\sum_{i=s+1}^{R} h_i(T)m_i \cong b - p(T) \qquad (24)$$

If we let H be the matrix whose i^{th} column is $h_i(T)$, $i = s+1, s+2, \ldots, R$, y be th column vector $b - p(T)$, and m be the column vector whose i^{th} component is m_i, then the above relation can be rewritten

$$Hm \cong y \ , \qquad (25)$$

where H is a matrix with R rows and R-s columns, and m and y are R-s dimensional column vectors. According to the theory of the method of least squares the vector m which minimizes the form

$$(Hm - y, Hm - y) = G \qquad (26)$$

is

$$m = (H'H)^{-1}H'y \ , \qquad (27)$$

where H′ is the transpose of H.

A few general comments on the procedure are now in order. The selection of a good initial approximation is important, since in this case the method is rapidly convergent, with the number of correct digits approximately doubling with each additional step; if, however, the initial approximation is too poor, the method may be divergent. At each step we must integrate R(R-s+1) first order differential equations with given initial conditions to produce the R-s homogeneous solutions and the one particular solution. In addition we must solve a system of R-s linear algebraic equations. This may be a sizable computing load, and the solving of the linear algebraic equations can be a source of great difficulty.(8)

V. NUMERICAL EXPERIMENTS TO DETERMINE c, THE THICKNESS OF THE LOWER LAYER

In the first series of experiments we wish to determine only the thickness of the lower layer, assuming that all of the other parameters of the medium are known. We use a seven-point Gaussian quadrature, so that $N = 7$. We consider c to be a function of τ_1 for which $dc/d\tau_1 = 0$. Following the method prescribed in the preceding section, we obtain $N^2 = 49$ linear differential equations for R_{ij}^{k+1} and one equation for c^{k+1}. Thus there are 50 linear differential equations

$$\begin{aligned}
\frac{dR_{ij}^{k+1}}{d\tau_1} = {} & \Big\{ -\left(\frac{1}{\mu_i} + \frac{1}{\mu_j}\right) R_{ij}^{k} + \lambda(c^k)\left(1 + \frac{1}{2}\sum_{\ell=1}^{N} R_{i\ell}^{k}\frac{W_\ell}{\mu_\ell}\right) \times \\
& \times \left(1 + \frac{1}{2}\sum_{\ell=1}^{N} R_{\ell j}^{k}\frac{W_\ell}{\mu_\ell}\right)\Big\} + \Big\{ -\left(\frac{1}{\mu_i} + \frac{1}{\mu_j}\right)\left(R_{ij}^{k+1} - R_{ij}^{k}\right) \\
& + \frac{1}{2}\lambda(c^k)\Big[\left(1 + \frac{1}{2}\sum_{\ell=1}^{N} R_{i\ell}^{k}\frac{W_\ell}{\mu_\ell}\right) \times \sum_{\ell=1}^{N}\left(R_{\ell j}^{k+1} - R_{\ell j}^{k}\right)\frac{W_\ell}{\mu_\ell} \\
& + \left(1 + \frac{1}{2}\sum_{\ell=1}^{N} R_{\ell j}^{k}\frac{W_\ell}{\mu_\ell}\right) \times \sum_{\ell=1}^{N}\left(R_{i\ell}^{k+1} - R_{i\ell}^{k}\right)\frac{W_\ell}{\mu_\ell}\Big]\Big\} \\
& + \Big\{(c^{k+1} - c^k)\left(1 + \frac{1}{2}\sum_{\ell=1}^{N} R_{i\ell}^{k}\frac{W_\ell}{\mu_\ell}\right)\left(1 + \frac{1}{2}\sum_{\ell=1}^{N} R_{\ell j}^{k}\frac{W_\ell}{\mu_\ell}\right) \times \\
& \times \left(-10\, b\ \text{sech}^2\, 10\,(\tau_1 - c^k)\right)\Big\}, \\
\frac{dc^{k+1}}{d\tau_1} = {} & 0,
\end{aligned} \tag{28}$$

where

$$\lambda(c^k) = a + b \tanh 10(\tau_1 - c^k)$$

One may reduce the number of differential equations in the system (28) by using the symmetry property

$$R_{ij}^{k+1}(\tau_1) = R_{ji}^{k+1}(\tau_1) , \qquad \begin{array}{l} i = 1,2,\ldots,N , \\ j = 1,2,\ldots,N . \end{array} \tag{29}$$

Thus instead of $N^2 + 1 = 50$ differential equations for R_{ij}^{k+1} and c^{k+1}, we have $(N(N+1)/2) + 1 = 29$ differential equations plus the finite Eqs. (29) to fill in the missing values of R. However, we will still speak of the matrix $\left\{R_{ij}^{k+1}\right\}$ as having 7 rows and 7 columns.

Now let the 50-dimensional vector $x^{k+1}(\tau_1)$ represent the 49 elements $R_{ij}^{k+1}(\tau_1)$ taken in some order,

$$x_\ell^{k+1}(\tau_1) = R_{ij}^{k+1}(\tau_1) \tag{30}$$

for $\ell = 1,2,\ldots,49$ as $i = 1,2,\ldots,7$, and $j = 1,2,\ldots,7$, and

$$x_{50}^{k+1}(\tau_1) = c^{k+1}(\tau_1) . \tag{31}$$

We express $x^{k+1}(\tau_1)$ as a sum of a particular solution p(t) and a homogeneous solution h(t) of the Eqs. (28),

$$x^{k+1}(\tau_1) = p(\tau_1) + m\, h(\tau_1) . \tag{32}$$

We require that the multiplier m be chosen to minimize the expression

$$\sum_{\ell=1}^{49} \left[p_\ell(1) + m\, h_\ell(1) - b_\ell\right]^2, \tag{33}$$

where the singly subscripted set $\{b_\ell\}$ ($\ell = 1,2,\ldots,49$) represents the doubly subscripted set of constants $\{4\mu_i b_{ij}\}$ ($i = 1,2,\ldots,7$, $j = 1,2,\ldots,7$) taken in the same order as the transformation (30). The value of m which minimizes (33) is that for which

$$m = \frac{\sum_{\ell=1}^{49} h_\ell(1)[b_\ell - p_\ell(1)]}{\sum_{\ell=1}^{49} [h_\ell(1)]^2}, \tag{34}$$

as we see from a simple differentiation. With the proper choice of initial values p(0) and h(0), this constant m is the initial value

$$m = c^{k+1}(0)$$

and hence gives directly the thickness of Layer 1.

The results of three experiments with different initial guesses of the thickness c are shown in Table 2. The initial approximation is generated in each run by integrating the nonlinear Eqs. (5) with the value of c listed as approximation zero.

Table 2

SUCCESSIVE APPROXIMATIONS OF c

Approximation	Run 1	Run 2	Run 3
0	0.2	0.8	0.0
1	0.62	0.57	
2	0.5187	0.5024	No
3	0.500089	0.499970	convergence
4	0.499990	0.499991	
True Value	0.5	0.5	0.5

Even with 60 per cent errors in the initial approximations, the value of c is determined to one part in fifty thousand or 0.002 per cent in four iterations. The time required on the IBM 7044 is 1½ minutes per run, using an integration step size of 0.01 and the Adams-Moulton method. Making use of the symmetry of R, 58 linear differential equations have to be integrated for the particular and homogeneous solutions for each approximation. In Run 3 the solution diverges because the initial guess is not good enough.

VI. NUMERICAL EXPERIMENTS TO DETERMINE T, THE OPTICAL THICKNESS

Next we consider the estimation of the total optical thickness T when all of the other system constants are known and 49 measurements {b} are given. The formulation of the linear boundary-value problem proceeds as for the previous case with one major difference. The terminal boundary for the interval of integration is unknown. In order that the interval be fixed, we define a new independent variable σ,

$$\sigma T = \tau_1 ,$$

with $0 \leq \sigma \leq 1$ as the range of integration. The constant T is a solution of the equation $dT/d\sigma = 0$. Then one may obtain the linear equations which correspond to (28), with independent variable σ instead of τ_1.

Three trials are made to determine T, with initial guesses $T^o = 0.9$, 1.5 and 0.5 respectively. Recall that the true value is $T = 1.0$. Within four minutes of computing time, four iterations are carried out per trial, and the results obtained are correct to one part in 100,000.

VII. NUMERICAL EXPERIMENTS TO DETERMINE λ_1, λ_2, and c, THE ALBEDOS AND OPTICAL THICKNESS

In the final experiment, on the basis of the 49 observations, we try to determine λ_1, λ_2, and c assuming T = 1.0 is known. Again we use N = 7. This time there are three homogeneous solutions and a particular solution to compute, each with $N(N+1)/2 + 3 = 31$ components, so that 4 x 31 = 124 linear differential equations are to be integrated during each stage of the successive approximation calculations. A standard (Gaussian elimination) matrix inversion procedure is used to invert the 3 x 3 matrix of the linear algebraic system. The results in Table 3 are obtained in three iterations which consume 2 minutes of computing time on the IBM 7044 machine.

Table 3

SUCCESSIVE APPROXIMATIONS OF λ_1, λ_2, AND c

Approximation	λ_1	λ_2	c
0	0.51	0.69	0.4
1	0.4200	0.6052	0.5038
2	0.399929	0.599995	0.499602
3	0.399938	0.599994	0.499878
True Values	0.4	0.6	0.5

VIII. DISCUSSION

The general conceptual and computational approach to inverse problems which we have discussed here is by no means limited to the simple physical model considered. At the expense of additional computing time we may treat atmospheres having anisotropic scattering, time-dependent sources, and so on. We may also wish to consider wave rather than particle processes. Applications to orbit determination,[11] system identification,[12,13] cardiology[14] and other areas[15] have been made. Of particular importance is the question of the effect of errors in the observations on the accuracy of the estimates of the parameters. In this connection it appears that the use of the min-max criterion rather than that of least squares may be efficacious. These and other matters will be discussed in later papers.

APPENDIX

This Appendix lists the FORTRAN IV programs which were written for (I) The Estimation of c, (II) The Estimation of T, and (III) The Estimation of λ_1, λ_2, and c. The major difference in the three programs lies in the subroutine DAUX, in which the right-hand sides of the differential equations being integrated must be evaluated in the same order as the dependent variables themselves and both sets stored in the array called "T". T(2) contains the current value of the independent variable, T(3) contains the integration step length, the next M locations from T(4) through T(M+3) contain the M variables which are being integrated, followed by the M derivatives in locations T(M+4) through T(2M+3). The integration routine used (RAND 7044 Library Routine Number W031) is called upon with the statements CALL INTS (T,M,2,0,0,0,0,0,0) and CALL INTM, and these in turn call upon subroutine DAUX.

In program (III), the matrix inversion and linear algebraic equation solving subroutine MATINV (RAND Library Routine Number W019) is used. The matrix of coefficients in the program is EMAT, the number of unknowns is three, and the right-hand vector is FVEC.

PROGRAM FOR THE ESTIMATION OF c, THE THICKNESS OF THE LOWER LAYER

```
$IBFTC RTINV
      COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPPNT,M1MAX,KMAX,DELTA,XTAU,
     1  ZERLAM,XLAM(2),B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
     2  P(28,101),H(28,3,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,7),
     3  H2(7,7,3),CONST(3),NEQ
C
C            PHASE I
C
    1 READ1000,N
      PRINT899
      PRINT900,N
      READ1001,(RT(I),I=1,N)
      PRINT901,(RT(I),I=1,N)
      READ1001,(WT(I),I=1,N)
      PRINT901,(WT(I),I=1,N)
      DO 2 I=1,N
      WR(I)=WT(I)/RT(I)
      DO 2 J=1,N
    2 AR(I,J)= 1.0/RT(I) + 1.0/RT(J)
C
  899 FORMAT(1H146X36HRADIATIVE TRANSFER - INVERSE PROBLEM / )
 1000 FORMAT(6I12)
  900 FORMAT(6I20)
 1001 FORMAT(6E12.8)
  901 FORMAT(6E20.8)
      READ1000,NPRNT,M1MAX,KMAX
      PRINT900,NPRNT,M1MAX,KMAX
      READ1001,DELTA
      PRINT901,DELTA
      READ1001,XTAU,ZERLAM,XLAM(1),XLAM(2)
      PRINT902
      PRINT903,XTAU,ZERLAM,XLAM(1),XLAM(2)
  902 FORMAT(1H123HPHASE I = TRUE SOLUTION /)
  903 FORMAT(1H0/
     1     1X11HTHICKNESS =, F10.4 /
     2     1X11HALBEDO(X) =, 20HA + B*TANH(10*(X-C))   //
     3  1X3HA =, E16.8, 10X3HB =, E16.8, 10X3HC =, E16.8 //)
      CALL NONLIN
      DO 3 I=1,N
      DO 3 J=1,N
    3 B2(I,J)=R2(I,J)
C
C
C            PHASE II
C
    4 READ1001,XTAU,ZERLAM,XLAM(1),XLAM(2)
      K=0
      PRINT904,K
      PRINT903,XTAU,ZERLAM,XLAM(1),XLAM(2)
C
      CALL NONLIN
C
  904 FORMAT(1H1   13HAPPROXIMATION, I3/ )
C
C            QUASILINEARIZATION ITERATIONS
```

```
C
      DO 5 K1=1,KMAX
      PRINT904,K1
      CALL PANDH
      CALL LINEAR
    5 CONTINUE
C
C
C
      READ1000,IGO
      GO TO (1,4),IGO
      END
$IBFTC DAUX
      SUBROUTINE DAUX
      DIMENSION V2(7,7),X(3),F(7),G(7)
      COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
     1  ZERLAM,XLAM(2),B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
     2  P(28,101),H(28,3,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,7),
     3  H2(7,7,3),CONST(3),NEQ
      GO TO (1,2),IFLAG
C
CNONLINEAR
C
    1 L=3
      DO 4 I=1,N
      DO 4 J=1,I
      L=L+1
    4 V2(I,J)=T(L)
      DO 5 I=1,N
      DO 5 J=I,N
    5 V2(I,J)=V2(J,I)
      L=L+1
      VLAM2=T(L)
      SIG=T(2)
      Y=XTAU*SIG
      X(1)=ZERLAM
      X(2)=XLAM(1)
      X(3)=VLAM2
      CALL ALBEDO(Y,X,Z)
      ZLAMDA=Z
C
      DO 6 I=1,N
      F(I)=0.0
      DO 7 K=1,N
    7 F(I)=F(I) + WR(K)*V2(I,K)
    6 F(I)=0.5*F(I) + 1.0
C
      DO 8 I=1,N
      DO 8 J=1,I
      L=L+1
      DR=-AR(I,J)*V2(I,J) + ZLAMDA*F(I)*F(J)
    8 T(L)=DR
      DO 9 I=1,1
      L=L+1
    9 T(L)=0.0
```

```
      RETURN
C
C
CLINEAR
C
   2  SIG=T(2)
      Y=XTAU*SIG
      X(1)=ZERLAM
      X(2)=XLAM(1)
      X(3)=XLAM(2)
      CALL ALBEDC(Y,X,Z)
      ZLAMDA=Z
C
      DO 16 I=1,N
      F(I)=0.0
      DO 17 K=1,N
  17  F(I)=F(I) + WR(K)*R2(I,K)
  16  F(I)=0.5*F(I) + 1.0
C
CP'S
C
      L=3
      DO 14 I=1,N
      DO 14 J=1,I
      L=L+1
  14  V2(I,J)=T(L)
      DO 15 I=1,N
      DO 15 J=I,N
  15  V2(I,J)=V2(J,I)
      L=L+1
      VLAM2=T(L)
C
      DO 10 I=1,N
      G(I)=0.0
      DO 10 K=1,N
  10  G(I)=G(I) + (V2(I,K)-R2(I,K))*WR(K)
      ARG=10.0*(Y-XLAM(2))
      XTANX=-10.0*XLAM(1)*(1.0-(TANH(ARG))**2)
        M=3+NEQ
      DO 12 I=1,N
      DO 12 J=1,I
      FIJ=F(I)*F(J)
      CAPF=-AR(I,J)*R2(I,J) + ZLAMDA*FIJ
      T1=     -AR(I,J)*(V2(I,J)-R2(I,J))
      T2=     0.5*ZLAMDA*(F(I)*G(J)+F(J)*G(I))
      T3=     CAPF
      T4=(VLAM2-XLAM(2))*XTANX*FIJ
        M=M+1
  12  T(M)=T1+T2+T3+T4
      DO 19 I=1,1
        M=M+1
  19  T(M)=0.0
C
CH'S
C
```

```
      DO 100 K=1,1
C
      DO 24 I=1,N
      DO 24 J=1,I
      L=L+1
   24 V2(I,J)=T(L)
      DO 25 I=1,N
      DO 25 J=I,N
   25 V2(I,J)=V2(J,I)
      L=L+1
      VLAM2=T(L)
C
      DO 20 I=1,N
      G(I)=0.0
      DO 20 J=1,N
   20 G(I)=G(I) +   V2(I,J)*WR(J)
C
      DO 22 I=1,N
      DO 22 J=1,I
      FIJ=F(I)*F(J)
      T1=      -AR(I,J)*V2(I,J)
      T2=      0.5*ZLAMDA*(F(I)*G(J)+F(J)*G(I))
      T3=0.0
      T4=VLAM2*XTANX*FIJ
       M=M+1
   22 T(M)=T1+T2+T3+T4
C
      DO 29 I=1,1
       M=M+1
   29 T(M)=0.0
  100 CONTINUE
      RETURN
      END
$IBFTC NONLIN
      SUBROUTINE NONLIN
      COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
     1 ZERLAM,XLAM(2),B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
     2 P(28,101),H(28,3,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,7),
     3 H2(7,7,3),CONST(3),NEQ
C
C           NONLINEAR D.E. FOP TRUE SOLUTION OR FOR INITIAL APPROX.
      IFLAG=1
      T(2)=0.0
      T(3)=DELTA
      M=1
      L1=0
      L3=3
      DO 1 I=1,N
      DO 1 J=1,I
      L1=L1+1
      L3=L3+1
      R2(I,J)=0.0
      R(L1,M)=R2(I,J)
    1 T(L3)=R2(I,J)
      L3=L3+1
    2  T(L3)=XLAM(2)
C
       NEQ=(N*(N+1))/2 + 1
       CALL INTS(T,NEQ,2,0,0,0,0,0,0)
C
       SIG=T(2)
       CALL OUTPUT
C
       DO 5 M1=1,M1MAX
       DO 4 M2=1,NPRNT
       CALL INTM
       M=M+1
       L1=0
       L3=3
       DO 3 I=1,N
       DO 3 J=1,I
       L1=L1+1
       L3=L3+1
       R2(I,J)=T(L3)
    3  R(L1,M)=R2(I,J)
    4  SIG=T(2)
    5  CALL OUTPUT
C
       RETURN
       END
$IBFTC PANDH
       SUBROUTINE PANDH
       COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
      1  ZERLAM,XLAM(2),B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
      2  P(28,101),H(28,3,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,7),
      3  H2(7,7,3),CONST(3),NEQ
       IFLAG=2
       T(2)=0.0
       T(3)=DELTA
       M=1
C P'S
C
       L1=0
       L3=3
       DO 1 I=1,N
       DO 1 J=1,I
       L1=L1+1
       L3=L3+1
       P(L1,M)=0.0
    1  T(L3)=P(L1,M)
       L3=L3+1
       PLAM(2)=0.0
    2  T(L3)=PLAM(2)
C
C H'S
C
       DO 7 K=1,1
       L1=0
       DO 3 I=1,N
       DO 3 J=1,I
      L1=L1+1
      L3=L3+1
      H(L1,K,M)=0.0
    3 T(L3)=H(L1,K,M)
C
      L3=L3+1
    6 HLAM(2,K)=1.0
    7 T(L3)=HLAM(2,K)
C
      L=0
      DO 8 I=1,N
      DO 8 J=1,I
      L=L+1
    8 R2(I,J)=R(L,M)
      DO 9 I=1,N
      DO 9 J=I,N
    9 R2(I,J)=R2(J,I)
C
      NEQ=2*((N*(N+1))/2 + 1)
      CALL INTS(T,NEQ,2.0,0,0,0,0,0)
      LMAX=(N*(N+1))/2
      PRINT52,T(2),(P(L,M),H(L,1,M),L=1,LMAX)
   52 FORMAT(1H0F9.4,5E20.8/(10X5E20.8))
C
      DO 51 M1=1,M1MAX
      DO 50 M2=1,NPRNT
      CALL INTM
      M=M+1
CPREV.APPROX. R(I,J)
      L1=0
      DO 10 I=1,N
      DO 10 J=1,I
      L1=L1+1
   10 R2(I,J)=R(L1,M)
      DO 11 I=1,N
      DO 11 J=I,N
   11 R2(I,J)=R2(J,I)
      L1=0
      L3=3
      DO 12 I=1,N
      DO 12 J=1,I
      L1=L1+1
      L3=L3+1
   12 P(L1,M)=T(L3)
      L3=L3+1
      DO 13 K=1,1
      L1=0
      DO 14 I=1,N
      DO 14 J=1,I
      L1=L1+1
      L3=L3+1
   14 H(L1,K,M)=T(L3)
   13 L3=L3+1
   50 CONTINUE
   51 PRINT52,T(2),(P(L,M),H(L,1,M),L=1,LMAX)
```

```
      RETURN
      END
$IBFTC LINEAR
      SUBROUTINE LINEAR
      DIMENSION CHKI(3)
      DIMENSION A(49,3),B(49),EMAT(50,50),          PIVOT(50),INDEX(50,2)
     1,IPIVOT(50),FVEC(50,1)
      COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
     1  ZERLAM,XLAM(2),B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
     2  P(28,101),H(28,3,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7 7),
     3  H2(7,7,3),CONST(3),NEQ
CBOUNDARY CONDITIONS
      MLAST=NPRNT*M1MAX + 1
      DO 1 K=1,1
      L=0
      DO 2 I=1,N
      DO 2 J=1,I
      L=L+1
    2 H2(I,J,K)=H(L,K,MLAST)
      DO 1 I=1,N
      DO 1 J=I,N
    1 H2(I,J,K)=H2(J,I,K)
      L=0
      DO 3 I=1,N
      DO 3 J=1,I
      L=L+1
    3 P2(I,J)=P(L,MLAST)
      DO 4 I=1,N
      DO 4 J=I,N
    4 P2(I,J)=P2(J,I)
CLEAST SQUARES
      DO 5 K=1,1
      L=C
      DO 5 I=1,N
      DO 5 J=1,N
      L=L+1
    5 A(L,K)=H2(I,J,K)
      L=0
      DO 6 I=1,N
      DO 6 J=1,N
      L=L+1
    6 B(L)=B2(I,J) - P2(I,J)
C
      LMAX=N**2
      PRINT60
   60 FORMAT(1H0)
      DO 61 L=1,LMAX
   61 PRINT82,(A(L,K),K=1,1),B(L)
C
      DO 8 I=1,1
      DO 7 J=1,1
      SUM=0.0
      DO 9 L=1,LMAX
    9 SUM=SUM + A(L,I)*A(L,J)
    7 EMAT(I,J)=SUM
```

```
      SUM=0.0
      DO 10 L=1,LMAX
   10 SUM=SUM + A(L,I)*B(L)
    8 FVEC(I,1)=SUM
C
      PRINT60
      DO 81 I=1,1
   81 PRINT82,(EMAT(I,J),J=1,1),FVEC(I,1)
   82 FORMAT(10X6E20.8)
C
      FVEC(1,1)=FVEC(1,1)/EMAT(1,1)
C
      DO 11 I=1,1
   11 CONST(I)=FVEC(I,1)
C
      XLAM(2)=CONST(1)
      PRINT903,XTAU,ZERLAM,XLAM(1),XLAM(2)
  903 FORMAT(1H0/
     1    1X11HTHICKNESS =, F10.4 /
     2    1X11HALBEDO(X) =, 20HA + B*TANH(10*(X-C))  //
     3 1X3HA =, E16.8, 10X3HB =, E16.8, 10X3HC =, E16.8 //)
C
CNEW APPROXIMATION
C
      M=1
      L=0
      DO 12 I=1,N
      DO 12 J=1,I
      L=L+1
      SUM=P(L,M)
      DO 13 K=1,1
   13 SUM =SUM + CONST(K)*H(L,K,M)
   12 R(L,M)=SUM
      L=0
      DO 14 I=1,N
      DO 14 J=1,I
      L=L+1
   14 R2(I,J)=R(L,M)
      SIG=0.0
      CALL OUTPUT
C
      DO 50 M1=1,M1MAX
      DO 18 M2=1,NPRNT
      M=M+1
      L=0
      DO 15 I=1,N
      DO 15 J=1,I
      L=L+1
      SUM=P(L,M)
      DO 16 K=1,1
   16 SUM=SUM + CONST(K)*H(L,K,M)
   15 R(L,M)=SUM
      L=0
      DO 17 I=1,N
      DO 17 J=1,I
      L=L+1
   17 R2(I,J)=R(L,M)
   18 SIG=SIG + DELTA
   50 CALL OUTPUT
C
      RETURN
      END
$IBFTC OUTPUT
      SUBROUTINE OUTPUT
      DIMENSION X(3)
      COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
     1  ZERLAM,XLAM(2),B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
     2  P(28,101),H(28,3,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,7),
     3  H2(7,7,3),CONST(3),NEQ
      DO 1 I=1,N
      DO 1 J=I,N
    1 R2(I,J)=R2(J,I)
      Y=XTAU*SIG
      X(1)=ZERLAM
      X(2)=XLAM(1)
      X(3)=XLAM(2)
      CALL ALBEDO(Y,X,Z)
      PRINT100, SIG,Y,Z
  100 FORMAT(1H0 7HSIGMA =,F6.2, 4X5HTAU =, F6.2, 4X8HALBEDO =,F6.2/)
      DO 2 J=1,N
    2 PRINT101,J,(R2(I,J),I=1,N)
  101 FORMAT(I10, 7F10.6)
      RETURN
      END
$IBFTC ALBEDO
      SUBROUTINE ALBEDO(Y,X,Z)
      DIMENSION X(3)
      COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
     1  ZERLAM,XLAM(2),B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
     2  P(28,101),H(28,3,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,7),
     3  H2(7,7,3),CONST(3),NEQ
      ARG=10.0*(Y-X(3))
      Z=X(1) + X(2)*TANH(ARG)
      RETURN
      END
```

II. PROGRAM FOR THE ESTIMATION OF T, THE THICKNESS OF THE MEDIUM

```
$IBFTC RTINV
      COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
     1  ZERLAM,XLAM(2),B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
     2  P(28,101),H(28,3,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,7),
     3  H2(7,7,3),CONST(3),NEQ
C
C              PHASE I
C
    1  READ1000,N
       PRINT899
       PRINT900,N
       READ1001,(RT(I),I=1,N)
       PRINT901,(RT(I),I=1,N)
       READ1001,(WT(I),I=1,N)
       PRINT901,(WT(I),I=1,N)
       DO 2 I=1,N
       WR(I)=WT(I)/RT(I)
       DO 2 J=1,N
    2  AR(I,J)= 1.0/RT(I) + 1.0/RT(J)
C
  899 FORMAT(1H146X36HRADIATIVE TRANSFER - INVERSE PROBLEM / )
 1000 FORMAT(6I12)
  900 FORMAT(6I20)
 1001 FORMAT(6E12.8)
  901 FORMAT(6E20.8)
      READ1000,NPRNT,M1MAX,KMAX
      PRINT900,NPRNT,M1MAX,KMAX
      READ1001,DELTA
      PRINT901,DELTA
      READ1001,XTAU,ZERLAM,XLAM(1),XLAM(2)
      PRINT902
      PRINT903,XTAU,ZERLAM,XLAM(1),XLAM(2)
  902 FORMAT(1H123HPHASE I - TRUE SOLUTION /)
  903 FORMAT(1H0/
     1    1X11HTHICKNESS =, F10.4 /
     2    1X11HALBEDO(X) =, 20HA + B*TANH(10*(X-C))  //
     3  1X3HA =, E16.8, 10X3HB =, E16.8, 10X3HC =, E16.8 //)
      CALL NONLIN
      DO 3 I=1,N
      DO 3 J=1,N
    3 B2(I,J)=R2(I,J)
C
C
C             PHASE II
C
    4 READ1001,XTAU,ZERLAM,XLAM(1),XLAM(2)
      K=0
      PRINT904,K
      PRINT903,XTAU,ZERLAM,XLAM(1),XLAM(2)
C
      CALL NONLIN
C
  904 FORMAT(1H1   13HAPPROXIMATION, I3/ )
C
C             QUASILINEARIZATION ITERATIONS
```

```
C
      DO 5 K1=1,KMAX
      PRINT904,K1
      CALL PANDH
      CALL LINEAR
    5 CONTINUE
C
C
C
      READ1000,IGO
      GO TO (1,4),IGO
      END
$IBFTC DAUX
      SUBROUTINE DAUX
      DIMENSION V2(7,7),X(3),F(7),G(7)
      COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
     1  ZERLAM,XLAM(2),B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
     2  P(28,101),H(28,3,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,7),
     3  H2(7,7,3),CONST(3),NEQ
      GO TO (1,2),IFLAG
C
CNONLINEAR
C
    1 L=3
      DO 4 I=1,N
      DO 4 J=1,I
      L=L+1
    4 V2(I,J)=T(L)
      DO 5 I=1,N
      DO 5 J=1,N
    5 V2(I,J)=V2(J,I)
      L=L+1
      VTAU=T(1)
      SIG=T(2)
      Y=VTAU*SIG
      X(1)=ZERLAM
      X(2)=XLAM(1)
      X(3)=XLAM(2)
      CALL ALBEDO(Y,X,Z)
      ZLAMDA=Z
C
      DO 6 I=1,N
      F(I)=0.0
      DO 7 K=1,N
    7 F(I)=F(I) + WR(K)*V2(I,K)
    6 F(I)=0.5*F(I) + 1.0
C
      DO 8 I=1,N
      DO 8 J=1,I
      L=L+1
      DR=-AR(I,J)*V2(I,J) + ZLAMDA*F(I)*F(J)
    8 T(L)=DR*VTAU
      DO 9 I=1,1
      L=L+1
    9 T(L)=0.0
```

```
      RETURN
C
C
CLINEAR
C
    2 SIG=T(2)
      Y=XTAU*SIG
      X(1)=ZERLAM
      X(2)=XLAM(1)
      X(3)=XLAM(2)
      CALL ALBEDO(Y,X,Z)
      ZLAMDA=Z
C
      DO 16 I=1,N
      F(I)=0.0
      DO 17 K=1,N
   17 F(I)=F(I) + WR(K)*R2(I,K)
   16 F(I)=0.5*F(I) + 1.0
C
CP'S
C
      L=3
      DO 14 I=1,N
      DO 14 J=1,I
      L=L+1
   14 V2(I,J)=T(L)
      DO 15 I=1,N
      DO 15 J=I,N
   15 V2(I,J)=V2(J,I)
      L=L+1
      VTAU=T(L)
C
      DO 10 I=1,N
      G(I)=0.0
      DO 10 K=1,N
   10 G(I)=G(I) + (V2(I,K)-R2(I,K))*WR(K)
      ARG=10.0*(Y-XLAM(2))
      PARTL=10.0*SIG*XLAM(1)*(1.0-(TANH(ARG))**2)
        M=3+NEQ
      DO 12 I=1,N
      DO 12 J=1,I
      FIJ=F(I)*F(J)
      CAPF=-AR(I,J)*R2(I,J) + ZLAMDA*FIJ
      T1=-XTAU*AR(I,J)*(V2(I,J)-R2(I,J))
      T2=XTAU*0.5*ZLAMDA*(F(I)*G(J)+F(J)*G(I))
      T3=XTAU*CAPF
      T4=(VTAU-XTAU)*(CAPF + XTAU*FIJ*PARTL)
        M=M+1
   12 T(M)=T1+T2+T3+T4
      DO 19 I=1,1
        M=M+1
   19 T(M)=0.0
C
CH'S
C
```

```
      DO 100 K=1,1
C
      DO 24 I=1,N
      DO 24 J=1,I
      L=L+1
   24 V2(I,J)=T(L)
      DO 25 I=1,N
      DO 25 J=I,N
   25 V2(I,J)=V2(J,I)
      L=L+1
      VTAU=T(L)
C
      DO 20 I=1,N
      G(I)=0.0
      DO 20 J=1,N
   20 G(I)=G(I) +  V2(I,J)*WR(J)
C
      DO 22 I=1,N
      DO 22 J=1,I
      FIJ=F(I)*F(J)
      CAPF=-AR(I,J)*R2(I,J) + ZLAMDA*FIJ
      T1=-XTAU*AR(I,J)*V2(I,J)
      T2=XTAU*0.5*ZLAMDA*(F(I)*G(J)+F(J)*G(I))
      T3=0.0
      T4=VTAU*(CAPF + XTAU*FIJ*PARTL)
        M=M+1
   22 T(M)=T1+T2+T3+T4
C
      DO 29 I=1,1
        M=M+1
   29 T(M)=0.0
  100 CONTINUE
      RETURN
      END
$IBFTC NONLIN
      SUBROUTINE NONLIN
      COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
     1  ZERLAM,XLAM(2),B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
     2  P(28,101),H(28,3,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,7),
     3  H2(7,7,3),CONST(3),NEQ
C          NONLINEAR D.E. FOR TRUE SOLUTION OR FOR INITIAL APPROX.
C
      IFLAG=1
      T(2)=0.0
      T(3)=DELTA
      M=1
      L1=0
      L3=3
      DO 1 I=1,N
      DO 1 J=1,I
      L1=L1+1
      L3=L3+1
      R2(I,J)=0.0
      R(L1,M)=R2(I,J)
    1 T(L3)=R2(I,J)
```

```
      L3=L3+1
    2 T(L3)=XTAU
C
      NEQ=(N*(N+1))/2 + 1
      CALL INTS(T,NEQ,2,0,0,0,0,0,0)
C
      SIG=T(2)
      CALL OUTPUT
C
      DO 5 M1=1,M1MAX
      DO 4 M2=1,NPRNT
      CALL INTM
      M=M+1
      L1=0
      L3=3
      DO 3 I=1,N
      DO 3 J=1,I
      L1=L1+1
      L3=L3+1
      R2(I,J)=T(L3)
    3 R(L1,M)=R2(I,J)
    4 SIG=T(2)
    5 CALL OUTPUT
C
      RETURN
      END
$IBFTC LINEAR
      SUBROUTINE LINEAR
      DIMENSION CHKI(3)
      DIMENSION A(49,3),B(49),EMAT(50,50),          PIVOT(50),INDEX(50,2)
     1,IPIVOT(50),FVEC(50,1)
      COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
     1  ZERLAM,XLAM(2),B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
     2  P(28,101),H(28,3,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,7),
     3  H2(7,7,3),CONST(3),NEQ
CBOUNDARY CONDITIONS
      MLAST=NPRNT*M1MAX + 1
      DO 1 K=1,1
      L=0
      DO 2 I=1,N
      DO 2 J=1,I
      L=L+1
    2 H2(I,J,K)=H(L,K,MLAST)
      DO 1 I=1,N
      DO 1 J=I,N
    1 H2(I,J,K)=H2(J,I,K)
      L=0
      DO 3 I=1,N
      DO 3 J=1,I
      L=L+1
    3 P2(I,J)=P(L,MLAST)
      DO 4 I=1,N
      DO 4 J=I,N
    4 P2(I,J)=P2(J,I)
CLEAST SQUARES
```

```
      DO 5 K=1,1
      L=0
      DO 5 I=1,N
      DO 5 J=1,N
      L=L+1
    5 A(L,K)=H2(I,J,K)
      L=0
      DO 6 I=1,N
      DO 6 J=1,N
      L=L+1
    6 B(L)=B2(I,J) - P2(I,J)
C
      LMAX=N**2
      PRINT60
   60 FORMAT(1H0)
      DO 61 L=1,LMAX
   61 PRINT82,(A(L,K),K=1,1),B(L)
C
      DO 8 I=1,1
      DO 7 J=1,1
      SUM=0.0
      DO 9 L=1,LMAX
    9 SUM=SUM + A(L,I)*A(L,J)
    7 EMAT(I,J)=SUM
      SUM=0.0
      DO 10 L=1,LMAX
   10 SUM=SUM + A(L,I)*B(L)
    8 FVEC(I,1)=SUM
C
      PRINT60
      DO 81 I=1,1
   81 PRINT82,(EMAT(I,J),J=1,1),FVEC(I,1)
   82 FORMAT(10X6E20.8)
C
      FVEC(1,1)=FVEC(1,1)/EMAT(1,1)
C
      DO 11 I=1,1
   11 CONST(I)=FVEC(I,1)
C
      XTAU   =CONST(1)
      PRINT903,XTAU,ZERLAM,XLAM(1),XLAM(2)
  903 FORMAT(1H0/
     1     1X11HTHICKNESS =, E16.8 /
     2     1X11HALBEDO(X) =, 20HA + B*TANH(10*(X-C))  //
     3  1X3HA =, E16.8, 10X3HB =, E16.8, 10X3HC =, E16.8 //)
C
CNEW APPROXIMATION
C
      M=1
      L=0
      DO 12 I=1,N
      DO 12 J=1,I
      L=L+1
      SUM=P(L,M)
      DO 13 K=1,1
   13 SUM =SUM + CONST(K)*H(L,K,M)
   12 R(L,M)=SUM
      L=0
      DO 14 I=1,N
      DO 14 J=1,I
      L=L+1
   14 R2(I,J)=R(L,M)
      SIG=0.0
      CALL OUTPUT
C
      DO 50 M1=1,M1MAX
      DO 18 M2=1,NPRNT
      M=M+1
      L=0
      DO 15 I=1,N
      DO 15 J=1,I
      L=L+1
      SUM=P(L,M)
      DO 16 K=1,1
   16 SUM=SUM + CONST(K)*H(L,K,M)
   15 R(L,M)=SUM
      L=0
      DO 17 I=1,N
      DO 17 J=1,I
      L=L+1
   17 R2(I,J)=R(L,M)
   18 SIG=SIG + DELTA
   50 CALL OUTPUT
C
      RETURN
      END
$IBFTC PANDH   LIST
      SUBROUTINE PANDH
      COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
     1  ZERLAM,XLAM(2),B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
     2  P(28,101),H(28,3,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,7),
     3  H2(7,7,3),CONST(3),NEQ
      IFLAG=2
      T(2)=0.0
      T(3)=DELTA
      M=1
C P'S
C
      L1=0
      L3=3
      DO 1 I=1,N
      DO 1 J=1,I
      L1=L1+1
      L3=L3+1
      P(L1,M)=0.0
    1 T(L3)=P(L1,M)
      L3=L3+1
      PTAU=0.0
    2 T(L3)=PTAU
C
```

```
C H'S
C
      DO 7 K=1,1
      L1=0
      DO 3 I=1,N
      DO 3 J=1,I
      L1=L1+1
      L3=L3+1
      H(L1,K,M)=0.0
    3 T(L3)=H(L1,K,M)
C
      L3=L3+1
    6 HTAU(K)=1.0
    7 T(L3)=HTAU(K)
C
      L=0
      DO 8 I=1,N
      DO 8 J=1,I
      L=L+1
    8 R2(I,J)=R(L,M)
      DO 9 I=1,N
      DO 9 J=I,N
    9 R2(I,J)=R2(J,I)
C
      NEQ=2*((N*(N+1))/2 + 1)
      CALL INTS(T,NEQ,2,0,0,0,0,0,0)
      LMAX=(N*(N+1))/2
C
      DO 51 M1=1,M1MAX
      DO 50 M2=1,NPRNT
      CALL INTM
      M=M+1
CPREV.APPROX. R(I,J)
      L1=0
      DO 10 I=1,N
      DO 10 J=1,I
      L1=L1+1
   10 R2(I,J)=R(L1,M)
      DO 11 I=1,N
      DO 11 J=I,N
   11 R2(I,J)=R2(J,I)
      L1=0
      L3=3
      DO 12 I=1,N
      DO 12 J=1,I
      L1=L1+1
      L3=L3+1
   12 P(L1,M)=T(L3)
      L3=L3+1
      DO 13 K=1,1
      L1=0
      DO 14 I=1,N
      DO 14 J=1,I
      L1=L1+1
      L3=L3+1
   14 H(L1,K,M)=T(L3)
   13 L3=L3+1
   50 CONTINUE
   51 CONTINUE
      RETURN
      END
$IBFTC OUTPUT
      SUBROUTINE OUTPUT
      DIMENSION X(3)
      COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
     1  ZERLAM,XLAM(2),B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
     2  P(28,101),H(28,3,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,7),
     3  H2(7,7,3),CONST(3),NEQ
      DO 1 I=1,N
      DO 1 J=I,N
    1 R2(I,J)=R2(J,I)
      Y=XTAU*SIG
      X(1)=ZERLAM
      X(2)=XLAM(1)
      X(3)=XLAM(2)
      CALL ALBEDO(Y,X,Z)
      PRINT100, SIG,Y,Z
  100 FORMAT(1H0 7HSIGMA =,F6.2, 4X5HTAU =, F6.2, 4X8HALBEDO =,F6.2/)
      DO 2 J=1,N
    2 PRINT101,J,(R2(I,J),I=1,N)
  101 FORMAT(I10, 7F10.6)
      RETURN
      END
$IBFTC ALBEDO
      SUBROUTINE ALBEDO(Y,X,Z)
      DIMENSION X(3)
      COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
     1  ZERLAM,XLAM(2),B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
     2  P(28,101),H(28,3,101),PTAU,PLAM(2),HTAU(3),HLAM(2,3),P2(7,7),
     3  H2(7,7,3),CONST(3),NEQ
      ARG=10.0*(Y-X(3))
      Z=X(1) + X(2)*TANH(ARG)
      RETURN
      END
```

III. PROGRAM FOR THE ESTIMATION OF λ_1, λ_2, AND c, THE TWO ALBEDOS AND THE THICKNESS OF THE LOWER LAYER

```
$IBFTC RTINV
      COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
     1  XLAM(3),       B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
     2  P(28,101),H(28,3,101),PLAM(3),HLAM(3,3),P2(7,7),
     3  H2(7,7,3),CONST(3),NEQ
C
C           PHASE I
C
   1  READ1000,N
      PRINT899
      PRINT900,N
      READ1001,(RT(I),I=1,N)
      PRINT901,(RT(I),I=1,N)
      READ1001,(WT(I),I=1,N)
      PRINT901,(WT(I),I=1,N)
      DO 2 I=1,N
      WR(I)=WT(I)/RT(I)
      DO 2 J=1,N
   2  AR(I,J)= 1.0/RT(I) + 1.0/RT(J)
C
  899 FORMAT(1H146X36HRADIATIVE TRANSFER - INDERSE PROBLEM / )
 1000 FORMAT(6I12)
  900 FORMAT(6I20)
 1001 FORMAT(6E12.8)
  901 FORMAT(6E20.8)
      READ1000,NPRNT,M1MAX,KMAX
      PRINT900,NPRNT,M1MAX,KMAX
      READ1001,DELTA
      PRINT901,DELTA
      READ1001,XTAU,(XLAM(I),I=1,3)
      PRINT902
      PRINT903,XTAU,(XLAM(I),I=1,3)
  902 FORMAT(1H123HPHASE I - TRUE SOLUTION /)
  903 FORMAT(1H0/
     1    1X11HTHICKNESS =, F10.4 /
     2    1X11HALBEDO(X) =, 20HA + B*TANH(10*(X-C))  //
     3  1X3HA =, E16.8, 10X3HB =, E16.8, 10X3HC =, E16.8 //)
      CALL NONLIN
      DO 3 I=1,N
      DO 3 J=1,N
   3  B2(I,J)=R2(I,J)
C
C
C           PHASE II
C
   4  READ1001,XTAU,(XLAM(I),I=1,3)
      K=0
      PRINT904,K
      PRINT903,XTAU,(XLAM(I),I=1,3)
C
      CALL NONLIN
C
  904 FORMAT(1H1   13HAPPROXIMATION, I3/ )
C
C           QUASILINEARIZATION ITERATIONS
```

```
C
      DO 5 K1=1,KMAX
      PRINT904,K1
      CALL PANDH
      CALL LINEAR
    5 CONTINUE
C
C
C
      READ1000,IGO
      GO TO (1,4),IGO
      END
$IBFTC DAUX     LIST
      SUBROUTINE DAUX
      DIMENSION V2(7,7),X(3),F(7),G(7)
     1   ,VLAM(3)
      COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
     1   XLAM(3),        B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
     2   P(28,101),H(28,3,101),PLAM(3),HLAM(3,3),P2(7,7),
     3   H2(7,7,3),CONST(3),NEQ
      GO TO (1,2),IFLAG
C
CNONLINEAR
C
    1 L=3
      DO 4 I=1,N
      DO 4 J=1,I
      L=L+1
    4 V2(I,J)=T(L)
      DO 5 I=1,N
      DO 5 J=I,N
    5 V2(I,J)=V2(J,I)
      DO 51 I=1,3
      L=L+1
   51 VLAM(I)=T(L)
      SIG=T(2)
      Y=XTAU*SIG
      DO 52 I=1,3
   52 X(I)=VLAM(I)
      CALL ALBEDO(Y,X,Z)
      ZLAMDA=Z
C
      DO 6 I=1,N
      F(I)=0.0
      DO 7 K=1,N
    7 F(I)=F(I) + WR(K)*V2(I,K)
    6 F(I)=0.5*F(I) + 1.0
C
      DO 8 I=1,N
      DO 8 J=1,I
      L=L+1
      DR=-AR(I,J)*V2(I,J) + ZLAMDA*F(I)*F(J)
    8 T(L)=DR
      DO 9 I=1,3
      L=L+1
    9 T(L)=0.0
      RETURN
C
C
CLINEAR
C
    2 SIG=T(2)
      Y=XTAU*SIG
      DO 21 I=1,3
   21 X(I)=XLAM(I)
      CALL ALBEDO(Y,X,Z)
      ZLAMDA=Z
C
      DO 16 I=1,N
      F(I)=0.0
      DO 17 K=1,N
   17 F(I)=F(I) + WR(K)*R2(I,K)
   16 F(I)=0.5*F(I) + 1.0
C
CP'S
C
      L=3
      DO 14 I=1,N
      DO 14 J=1,I
      L=L+1
   14 V2(I,J)=T(L)
      DO 15 I=1,N
      DO 15 J=I,N
   15 V2(I,J)=V2(J,I)
      DO 18 I=1,3
      L=L+1
   18 VLAM(I)=T(L)
C
      DO 10 I=1,N
      G(I)=0.0
      DO 10 K=1,N
   10 G(I)=G(I) + (V2(I,K)-R2(I,K))*WR(K)
      ARG=10.0*(Y-XLAM(3))
      TARG=TANH(ARG)
      XTANX=-10.0*XLAM(2)*(1.0-TARG**2)
        M=3+NEQ
      DO 12 I=1,N
      DO 12 J=1,I
      FIJ=F(I)*F(J)
      CAPF=-AR(I,J)*R2(I,J) + ZLAMDA*FIJ
      T1=CAPF
      T2=-AR(I,J)*(V2(I,J)-R2(I,J))
     1  + 0.5*ZLAMDA*(F(I)*G(J) + F(J)*G(I))
      T3=(VLAM(1)-XLAM(1))*FIJ
      T4=(VLAM(2)-XLAM(2))*TARG*FIJ
      T5=(VLAM(3)-XLAM(3))*XTANX*FIJ
        M=M+1
   12 T(M)=T1+T2+T3+T4+T5
      DO 19 I=1,3
        M=M+1
   19  T(M)=0.0
 C
 CH'S
 C
       DO 100 K= ,3
 C
       DO 24 I=1,N
       DO 24 J=1,I
       L=L+1
   24  V2(I,J)=T(L)
       DO 25 I=1,N
       DO 25 J=I,N
   25  V2(I,J)=V2(J,I)
       DO 26 I=1,3
       L=L+1
   26  VLAM(I)=T(L)
 C
       DO 20 I=1,N
       G(I)=0.0
       DO 20 J=1,N
   20  G(I)=G(I) +  V2(I,J)*WR(J)
 C
       DO 22 I=1,N
       DO 22 J=1,I
       FIJ=F(I)*F(J)
       T1=0.0
       T2=-AR(I,J)*V2(I,J) + 0.5*ZLAMDA*(F(I)*G(J) + F(J)*G(I))
       T3=VLAM(1)*FIJ
       T4=VLAM(2)*TARG*FIJ
       T5=VLAM(3)*XTANX*FIJ
         M=M+1
   22  T(M)=T1+T2+T3+T4+T5
 C
       DO 29 I=1,3
         M=M+1
   29  T(M)=0.0
  100  CONTINUE
       RETURN
       END
$IBFTC NONLIN
       SUBROUTINE NONLIN
       COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
      1 XLAM(3),        B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
      2 P(28,101),H(28,3,101),PLAM(3),HLAM(3,3),P2(7,7),
      3 H2(7,7,3),CONST(3),NEQ
C           NONLINEAR D.E. FOR TRUE SOLUTION OR FOR INITIAL APPROX.
C
       IFLAG=1
       T(2)=0.0
       T(3)=DELTA
       M=1
       L1=0
       L3=3
       DO 1 I=1,N
       DO 1 J=1,I
      L1=L1+1
      L3=L3+1
      R2(I,J)=0.0
      R(L1,M)=R2(I,J)
    1 T(L3)=R2(I,J)
      DO 2 I=1,3
      L3=L3+1
    2 T(L3)=XLAM(I)
C
      NEQ=(N*(N+1))/2 + 3
      CALL INTS(T,NEQ,2,0,0,0,0,0,0)
C
      SIG=T(2)
      CALL OUTPUT
C
      DO 5 M1=1,M1MAX
      DO 4 M2=1,NPRNT
      CALL INTM
      M=M+1
      L1=0
      L3=3
      DO 3 I=1,N
      DO 3 J=1,I
      L1=L1+1
      L3=L3+1
      R2(I,J)=T(L3)
    3 R(L1,M)=R2(I,J)
    4 SIG=T(2)
    5 CALL OUTPUT
C
      RETURN
      END
$IBFTC PANDH
      SUBROUTINE PANDH
      COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
     1  XLAM(3),         E2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
     2  F(28,101),H(28,3,101),PLAM(3),HLAM(3,3),P2(7,7),
     3  H2(7,7,3),CONST(2),NEQ
      IFLAG=2
      T(2)=0.0
      T(3)=DELTA
      M=1
C P'S
C
      L1=0
      L3=3
      DO 1 I=1,N
      DO 1 J=1,I
      L1=L1+1
      L3=L3+1
      P(L1,M)=0.0
    1 T(L3)=P(L1,M)
      DO 2 I=1,3
      L3=L3+1
      PLAM(I)=0.0
    2   T(L3)=PLAM(I)
C
C H'S
C
        DO 7 K=1,3
        L1=0
        DO 3 I=1,N
        DO 3 J=1,I
        L1=L1+1
        L3=L3+1
        H(L1,K,M)=0.0
    3   T(L3)=H(L1,K,M)
C
        DO 7 I=1,3
        L3=L3+1
        HLAM(I,K)=0.0
        IF(I-K)7,6,7
    6   HLAM(I,K)=1.0
    7   T(L3)=HLAM(I,K)
C
        L=0
        DO 8 I=1,N
        DO 8 J=1,I
        L=L+1
    8   R2(I,J)=R(L,M)
        DO 9 I=1,N
        DO 9 J=I,N
    9   R2(I,J)=R2(J,I)
C
        NEQ=4*((N*(N+1))/2 + 3)
        CALL INTS(T,NEQ,2,0,0,0,0,0,0)
        LMAX=(N*(N+1))/2
        PRINT52,T(2),(P(L,M),H(L,1,M),L=1,LMAX)
   52   FORMAT(1H0F9.4,5E20.8/(10X5E20.8))
C
        DO 51 M1=1,M1MAX
        DO 50 M2=1,NPRNT
        CALL INTM
        M=M+1
CPREV.APPROX. R(I,J)
        L1=0
        DO 10 I=1,N
        DO 10 J=1,I
        L1=L1+1
   10   R2(I,J)=R(L1,M)
        DO 11 I=1,N
        DO 11 J=I,N
   11   R2(I,J)=R2(J,I)
        L1=0
        L3=3
        DO 12 I=1,N
        DO 12 J=1,I
        L1=L1+1
        L3=L3+1
   12   P(L1,M)=T(L3)
```

```
      L3=L3+3
      DO 13 K=1,3
      L1=0
      DO 14 I=1,N
      DO 14 J=1,I
      L1=L1+1
      L3=L3+1
   14 H(L1,K,M)=T(L3)
   13 L3=L3+3
   50 CONTINUE
   51 PRINT52,T(2),(P(L,M),H(L,1,M),L=1,LMAX)
      RETURN
      END
$IBFTC LINEAR
      SUBROUTINE LINEAR
      DIMENSION CHKI(3)
      DIMENSION A(49,3),B(49),EMAT(50,50),          PIVOT(50),INDEX(50,2)
     1,IPIVOT(50),FVEC(50,1)
      COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
     1  XLAM(3),          B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
     2  P(28,101),H(28,3,101),PLAM(3),HLAM(3,3),P2(7,7),
     3  H2(7,7,3),CONST(3),NEQ
CBOUNDARY CONDITIONS
      MLAST=NPRNT*M1MAX + 1
      DO 1 K=1,3
      L=0
      DO 2 I=1,N
      DO 2 J=1,I
      L=L+1
    2 H2(I,J,K)=H(L,K,MLAST)
      DO 1 I=1,N
      DO 1 J=I,N
    1 H2(I,J,K)=H2(J,I,K)
      L=0
      DO 3 I=1,N
      DO 3 J=1,I
      L=L+1
    3 P2(I,J)=P(L,MLAST)
      DO 4 I=1,N
      DO 4 J=I,N
    4 P2(I,J)=P2(J,I)
CLEAST SQUARES
      DO 5 K=1,3
      L=0
      DO 5 I=1,N
      DO 5 J=1,N
      L=L+1
    5 A(L,K)=H2(I,J,K)
      L=0
      DO 6 I=1,N
      DO 6 J=1,N
      L=L+1
    6 B(L)=B2(I,J) - P2(I,J)
C
      LMAX=N**2
      PRINT60
   60 FORMAT(1H0)
      DO 61 L=1,LMAX
   61 PRINT82,(A(L,K),K=1,3),B(L)
C
      DO 8 I=1,3
      DO 7 J=1,3
      SUM=0.0
      DO 9 L=1,LMAX
    9 SUM=SUM + A(L,I)*A(L,J)
    7 EMAT(I,J)=SUM
      SUM=0.0
      DO 10 L=1,LMAX
   10 SUM=SUM + A(L,I)*B(L)
    8 FVEC(I,1)=SUM
C
      PRINT60
      DO 81 I=1,3
   81 PRINT82,(EMAT(I,J),J=1,3),FVEC(I,1)
   82 FORMAT(10X6E20.8)
C
      CALL MATINV(EMAT,3,FVEC,1,DETERM,PIVOT,INDEX,IPIVOT)
C
      DO 11 I=1,3
   11 CONST(I)=FVEC(I,1)
C
      DO 20 I=1,3
   20 XLAM(I)=CONST(I)
      PRINT903,XTAU,(XLAM(I),I=1,3)
  903 FORMAT(1H0/
     1 1X11HTHICKNESS =, E16.8 /
     2    1X11HALBEDO(X) =, 20HA + B*TANH(10*(X-C))  //
     3 1X3HA =, E16.8, 10X3HB =, E16.8, 10X3HC =, E16.8 //)
C
CNEW APPROXIMATION
C
      M=1
      L=0
      DO 12 I=1,N
      DO 12 J=1,I
      L=L+1
      SUM=P(L,M)
      DO 13 K=1,3
   13 SUM =SUM + CONST(K)*H(L,K,M)
   12 R(L,M)=SUM
      L=0
      DO 14 I=1,N
      DO 14 J=1,I
      L=L+1
   14 R2(I,J)=R(L,M)
      SIG=0.0
      CALL OUTPUT
C
      DO 50 M1=1,M1MAX
      DO [illegible] M2=1,NPRNT
      M=M+1
      L=0
      DO 15 I=1,N
      DO 15 J=1,I
      L=L+1
      SUM=P(L,M)
      DO 16 K=1,3
   16 SUM=SUM + CONST(K)*H(L,K,M)
   15 R(L,M)=SUM
      L=0
      DO 17 I=1,N
      DO 17 J=1,I
      L=L+1
   17 R2(I,J)=R(L,M)
   18 SIG=SIG + DELTA
   50 CALL OUTPUT
C
      RETURN
      END
$IBFTC OUTPUT
      SUBROUTINE OUTPUT
      DIMENSION X(3)
      COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
     1  XLAM(3),        B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
     2  P(28,101),H(28,3,101),PLAM(3),HLAM(3,3),P2(7,7),
     3  H2(7,7,3),CONST(3),NEQ
      DO 1 I=1,N
      DO 1 J=I,N
    1 R2(I,J)=R2(J,I)
      Y=XTAU*SIG
      DO 3 I=1,3
    3 X(I)=XLAM(I)
      CALL ALBEDO(Y,X,Z)
      PRINT100, SIG,Y,Z
  100 FORMAT(1H0 7HSIGMA =,F6.2, 4X5HTAU =, F6.2, 4X8HALBEDO =,F6.2/)
      DO 2 J=1,N
    2 PRINT101,J,(R2(I,J),I=1,N)
  101 FORMAT(I10, 7F10.6)
      RETURN
      END
$IBFTC ALBEDO
      SUBROUTINE ALBEDO(Y,X,Z)
      DIMENSION X(3)
      COMMON N,RT(7),WT(7),WR(7),AR(7,7),NPRNT,M1MAX,KMAX,DELTA,XTAU,
     1  XLAM(3),        B2(7,7),R2(7,7),IFLAG,R(28,101),T(1491),SIG,
     2  P(28,101),H(28,3,101),PLAM(3),HLAM(3,3),P2(7,7),
     3  H2(7,7,3),CONST(3),NEQ
      ARG=10.0*(Y-X(3))
      Z=X(1) + X(2)*TANH(ARG)
      RETURN
      END
```

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